RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2019

FIRST YEAR [BATCH 2019-22]

MATHEMATICS (Honours)

Paper : I (CC1)

Date : 11/12/2019

Time : 11 am – 1 pm

Full Marks : 50

(Use a separate Answer book for each group)

Group – A

Answer **any five** questions of the following : [5 × 5]

1. Let H and K be finite subgroups of a group G. Then show that $|HK| = \dfrac{|H||K|}{|H \cap K|}$.

2. Prove that a finite group G is a cyclic group iff there exists an element $a \in G$ such that $o(a) = |G|$.

3. Prove that a group G cannot be the union of two proper subgroups of G.

4. Show that the number of even permutations in $S_n, (n \geq 2)$ is the same as that of the odd permutations.

5. Let a, b be fixed positive integers and $H = \{ax + by : x, y \in \mathbb{Z}\}$. Show that H is a cyclic group.

6. Show that for any finite set A, if $f : A \to A$ is injective, then f^n is injective for all integers $n(\geq 1)$ and also show that f is surjective.

7. a) A relation R on the set of non-zero complex numbers is defined by xRy iff $\dfrac{x-y}{x+y}$ is real. Show that R is not an equivalence relation [4]

   b) Give an example of a relation which is both symmetric and anti-symmetric. [1]

8. a) Prove that the semigroup $(G, \circ)$ is a non-commutative group where $G = \{(a,b) \in \mathbb{Q} \times \mathbb{Q} : a \neq 0\}$ and the composition 'o' is defined by $(a,b) \circ (c,d) = (ac, ad + b)$ for $(a,b), (c,d)$ in G [$\mathbb{Q}$ in the set of rational numbers]. [3]

   b) Let G be an abelian group. Prove that the subset $H = \{g \in G : g^2 = e \text{ (idenity element)}\}$ forms a subgroup of G. [2]

Group – B

Answer **any five** questions of the following : [5 × 5]

9. If $x, y \in \mathbb{R}$ with $y > 0$, then prove that $\exists n \in \mathbb{N}$ such that $ny > x$. Hence deduce that $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}$ such that $n > x$. [4+1]

10. a) Prove that $\sup\{r \in Q : r < b\} = b$ for each $b \in \mathbb{R}$ where Q is the set of all rational numbers. [3]

    b) If A be a nonempty bounded subset of R then show that $A' \cap (b, \infty) = \varnothing$ where $b = \sup A$ and A' is the derived set of A. [2]

11. Prove that interior of any set in $\mathbb{R}$ is an open set. [5]

12. a) Let $G \subseteq \mathbb{R}$ be an open set and $F \subseteq \mathbb{R}$ be a closed set. Prove that $G - F$ is an open set and $F - G$ is a closed set. [3]

    b) Prove or disprove: ''Arbitrary intersection of open subsets of $\mathbb{R}$ is open''. [2]

13. Let $[a_n, b_n]$ be a closed and bounded interval in $\mathbb{R}\ \forall n \in \mathbb{N}$. Let $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \forall n \in \mathbb{N}$ and $\delta_n = b_n - a_n \to 0$ as $n \to \infty$. Prove that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is a singleton set in $\mathbb{R}$. [5]

14. a) If for a sequence $\{x_n\}$ of real numbers, $\lim_{n\to\infty} x_{3n-2} = \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} x_{3n} = l (\in \mathbb{R})$ then prove that $\{x_n\}$ is convergent. [3]

    b) Give an example of a sequence of real numbers with exactly three limit points. Justify your answer. [2]

15. Let $f : \mathbb{Q} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y);\ x, y \in \mathbb{Q}$. Then show that $f(x) = ax$ where $a = f(1)$. [5]

16. Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R},\ g : S \to \mathbb{R}$ be two functions. Let 'a' be a limit point of S and g is bounded in some $N'_\delta(a) \cap S$ and $\lim_{x\to a} f(x) = 0$. Then prove that $\lim_{x\to a} (f \cdot g)(x) = 0$. [5]

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